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Tuning the RADIO to the Off-Shell 2D Fayet Hypermultiplet Problem [1]

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ABSTRACT

We show via use of the RADIO technique that an off-shell (4,0) version of the hypermultiplet, in the form first proposed by Fayet, exists and contains 28 - 28 component fields. The off-shell structure uncovered is found to include a chiral truncation of the "generalized 2D, N = 4 tensor multiplet formalism" proposed by Ketov. The (4,0) theory is extended to an off-shell 56 - 56 component field (4,4) theory with the addition of a minimal (4,0) minus spinor multiplet together with (4,0) auxiliary multiplets. We propose that our final result gives a solution to a twenty year-old 2D supersymmetry problem in the physics literature.

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I. Introduction

Twenty years ago, P. Fayet [1] posed an intellectual "Gordian Knot" when he presented the on-shell hypermultiplet. This was the first example of a component level formulation which possesses 4D, N = 2 supersymmetry. The "knot" to be unraveled is the problem of finding a complete set of auxiliary fields to accompany the on-shell spectrum that he found. The supersymmetry algebra in his original formulation closes on some of the fields only with the use of equations of motion. The physical plus auxiliary fields would then allow an off-shell realization of the 4D, N = 2 supersymmetry algebra without the need to use equations of motion to close the algebra. A better known but similar problem is that of finding auxiliary fields for 4D, N = 4 supersymmetric Yang-Mills systems. Both of these are examples of the general "off-shell N-extended supersymmetry problem."

The off-shell problem for 4D, N = 1 supersymmetry was essentially solved by the introduction of Salam-Strathdee superspace and superfields [2]. For higher values of N, the problem remains unsolved because of the need to impose kinematic constraints[4] on the superfields in order to obtain irreducible or minimal off-shell representations. In fact, for a given off-shell representation, it is not even clear how many superfields are needed! Up until now, the choice of these kinematic constraints has been dependent on guesswork, luck, etc. as opposed to being a science. Within the last year however, we have seen hints from some developments within the study of 1D supersymmetric representation theory [3] that these required differential equations are actually determined by an algebraic structure we denote by $\mathcal{GR}(d, \mathrm{N})$. In particular, we have suggested that all supersymmetry representations (both on-shell and off-shell) are isomorphic to representations of this algebra. We have also found that there exist certain duality-like transformations that act on the representations of the algebra.

The off-shell supersymmetry problem even extends all the way to superstring and heterotic string theory [4]! In this last regard, it is conceivable that an increase in our understanding of the auxiliary fields may even help provide some insight into the fundamental problem facing covariant superstring field theory, namely "What is the stringy extension of the equivalence principle of general relativity?"

The situation with off-shell supersymmetry is highly unsatisfactory. One of the main reasons is that the auxiliary fields are closely related to the different types of dynamics to which the propagating fields are subject. This was recently illustrated with

[4] In a sense, kinematic constraints may be called Bianchi identities.

the example of the 4D, N = 1 WZNW term [5]. As well, this situation is frustrating. This has led to responses that can be placed into three categories; (a.) unreasonable proposals to go off-shell [6], (b.) theorems which "prove" off-shell representations suitable for actions are not possible with a finite number of auxiliary fields [7] and (c.) off-shell representations suitable for actions with an infinite number of auxiliary fields [8].

The last one of these permits the construction of off-shell actions that propagate the same physical degrees of freedom as the on-shell theory. Along these lines, we have long suspected that it should be possible to truncate these theories so that only a finite number of auxiliary fields will suffice to describe an off-shell theory. This was based on a little known observation of 4D, N = 1 scalar multiplet theory [9] where it was shown that it is possible to formulate the 4D, N = 1 scalar multiplet so that it requires an infinite number of auxiliary fields to describe an off-shell action. Although this is possible, it is terribly inconvenient to use such a formulation. After all, it is far simpler to use one of the off-shell representations that possess a finite number of auxiliary fields.

Until recently, the almost total lack of understanding of off-shell N-extended supersymmetry representations even extended down to the simplest ones, the spinning particles. Some time ago, we resolved to undertake a comprehensive study of the off-shell problem within the context of 1D theories [3]. The most interesting outcome of this study has been the new point of view it has provided on the off-shell supersymmetry problem. In fact, we have found both new on-shell representations [10] and off-shell representations [11] due to the use of new tools unearthed in our study. Perhaps the most unexpected of these new tools is the RADIO technique [10]. The RADIO technique permits the derivation of new on-shell and off-shell representations by starting from a D-dimensional, N-extended theory reducing (R) to 1D, performing certain "automorphic duality" (AD) transformations, integrating additional 1D representations (I) and then oxidizing (O) back up to a different D-dimensional, N-extended theory.

Although, Fayet's original puzzle has not been solved before, there does exists an off-shell 4D, N = 2 representation that should be closely related to the putative one for the hypermultiplet. Some time ago, it was shown that a "relaxed hypermultiplet" [12] exists. This is an excellent candidate upon which to apply the RADIO. This will be the purpose of this presentation. By the end of this present work we will propose that the 2D version of this puzzle has a surprisingly simple solution. There exists a 28 - 28 off-shell (4,0) hypermultiplet that possesses exactly the on-shell limit required by the Fayet's original hypermultiplet formulation. We will also find that there exists

a 28 - 28 off-shell (4,0) minus spinor multiplet[5] that has the correct structure so that when it is added to the (4,0) hypermultiplet, the two together describe an off-shell full (4,4) hypermultiplet with 56 - 56 components.

II. (4,4) Relaxed Hypermultiplet → RADIO → (4,0) Original Hypermultiplet

The work of Howe, Stelle and Townsend [12] can be recalled by introducing three real superfields (S, L_{ij}, L_{ijkl})

$$S = \overline{S} \quad , \quad L_{ij} = C_{ik}C_{jl}\overline{L}^{kl} \quad , \quad L_{ijkl} = C_{ir}C_{js}C_{kt}C_{lu}\overline{L}^{rstu} \quad , \tag{2.1}$$

that satisfy two types of differential equations. These are summarized in the table below,

Kinematic Constraints	Equations of Motion
$D_{\alpha(i\|}L_{\|jklm)} = 0$	$D_{\alpha i}S = i\frac{2}{3}C^{jk}D_{\alpha j}L_{ik}$
$D_{\alpha(i\|}L_{\|jk)} = C^{lm}D_{\alpha l}L_{ijkm}$	$L_{ijkl} = 0$
$C^{ij}D_{\alpha i}D_{\beta j}S = 0$	
$[\, D_{\alpha i} \, , \, \overline{D}_{\dot\beta}{}^{i} \,]S = 0$	

Table I

The kinematic constraints above are an example of the guesswork mentioned in the introduction. The equations of motion are derived from an action of the form

$$\mathcal{S}_{\rm RHM} = \int d^4x\, d^4\zeta\, d^4\bar\zeta \;\; [\, (\lambda_\alpha{}^i\rho^\alpha{}_i + \overline{\lambda}_{\dot\alpha i}\overline{\rho}^{\dot\alpha i}) + L^{ijkl}X_{ijkl} \,] \quad , \tag{2.2}$$

where variation with respect to the pre-potential $\rho^\alpha{}_i$ leads to the first equation of motion and variation with respect to X_{ijkl} leads to the second equation of motion. Finally, a complete enumeration of the component fields is given by

$$(L_{ij},\, L_{ijkl},\, \lambda_{\alpha i},\, \psi_{\alpha ijk},\, M_{ij},\, N,\, G_a,\, V_a{}_i{}^j,\, \xi_{\alpha i}\,) \quad , \tag{2.3}$$

for those contained in $\rho^\alpha{}_i$ and as well

$$(S,\, \psi_{\alpha i},\, K_{ij},\, A_a{}_i{}^j,\, \xi_{\alpha ijk},\, C_{ijkl}\,) \quad , \tag{2.4}$$

for those in X_{ijkl}. The engineering dimensions of these fields are given by

[5] There is a widespread misnomer regarding minus spinor multiplets. In the literature these are often called (0,4) multiplets. This is technically incorrect because the multiplets are still only representations of (4,0) supersymmetry.

Component Fields	2D Engineering Dimension
L_{ij}, L_{ijkl}, S	0
$\lambda_{\alpha i}$, $\psi_{\alpha i}$, $\psi_{\alpha ijk}$	1/2
K_{ij}, M_{ij}, N, G_a, $V_{a\,i}{}^j$, $A_{a\,i}{}^j$	1
$\xi_{\alpha i}$, $\xi_{\alpha ijk}$	3/2
C_{ijkl}	2

Table II

The first step of the RADIO is to reduce this to 1D. This will increase the number of supersymmetries to four. If we instead reduce to a 2D heterotic (4,0) formulation this is equivalent to the 1D theory. Under this reduction each of the pre-potentials splits into two distinct representations. We will call the first of these, obtained after the reduction, the 2D (4,0) RHM (relaxed hypermultiplet) theory. The component fields of the 2D (4,0) RHM representation are given by,

$$(L_{ij},\, L_{ijkl},\, \lambda_{+i},\, \psi_{+ijk},\, G_{\ddagger},\ G_{=},\, V_{\ddagger\,i}{}^j,\, V_{=\,i}{}^j,\, \xi_{-i}\ ) \quad , \tag{2.5}$$

and as well

$$(S,\, \psi_{+i},\, A_{\ddagger\,i}{}^j,\, A_{=\,i}{}^j,\, \xi_{-ijk},\, C_{ijkl}\ ) \quad . \tag{2.6}$$

We call the second representation a non-minimal minus spinor multiplet (NM^{2}SM). The component fields of the NM^{2}SM representation are

$$(\lambda_{-i},\, \psi_{-ijk},\, M_{ij},\, N,\, s,\, p,\, s_i{}^j,\ p_i{}^j,\, \xi_{+i}\ ) \quad , \tag{2.7}$$

and as well

$$(\psi_{-i},\, \widetilde{s}_i{}^j,\, \widetilde{p}_i{}^j,\, K_{ij},\, \xi_{+ijk}\ ) \quad . \tag{2.8}$$

If we start with G_a in four dimensions, under reduction to 2D we have $G_a \ \rightarrow\ (s,\, p,\, G_{\ddagger},\, G_{=}\ )$. The resulting fields split between the two 2D (4,0) multiplets. The two fields $(s,\, p\,)$ go to the NM^{2}SM theory. The two fields $(G_{\ddagger},\, G_{=}\ )$ go to the 2D (4,0) RHM theory. Similar observations apply to $A_{a\,i}{}^j$ and $V_{a\,i}{}^j$. One other point of convenience is to note that the components s and p can be combined into two complex fields $u = s\, +\, ip$ and $\overline{u} = s\, -\, ip$.

We next apply a Klein flip operator to the NM^{2}SM theory. The net effect of this operator is that it multiplies every field in (2.7) by a lower $+$-index and each field in (2.8) by a lower $-$-index. The Klein flip is one of a number of duality-type transformations that have been found to exist for 1D N-extended supersymmetric

theories [3]. It also has the interesting effect that it turns scalar multiplets into spinor multiplets and vice-versa. The multiplet we obtain after this AD map has a spectrum that consists of two sub-multiplets. The component fields of the first, with two field strengths $\mathcal{A}_i$ and $\mathcal{A}_{ijk}$, has the structure

$$(A_i,\, A_{ijk},\, \overline{\pi}^-,\, \rho^-,\, \overline{\lambda}^-{}_{ij},\, \rho^-{}_i{}^j,\, P_{\ddagger i}\,) \quad , \tag{2.9}$$

and the second has the field strength $\mathcal{B}_{=\, i}$ along with the component field content,

$$(P_{=\, i},\, \overline{\chi}^+{}_{ij},\, \beta^+{}_i{}^j,\, B_{ijk}\,) \quad . \tag{2.10}$$

All fields with more than one SU(2) index are totally symmetrical when those indices are at the same height. This is a consequence of applying the Klein flip operator to the RHM theory.

III. An Off-Shell Version of the (4,0) Original Hypermultiplet

The basic superfields that contain all of the component fields of the off-shell (4,0) version of the original hypermultiplet are $\mathcal{A}_i$, $\mathcal{A}_{ijk}$ and $\mathcal{B}_{=i}$. The spectrum of the fields in (2.9) and (2.10) imply that the superfields satisfy the kinematic constraints,

$$D_{+(i}\mathcal{A}_{j)} ~=~ C^{kl}D_{+k}\mathcal{A}_{ijl} \quad , \quad D_{+(i}\mathcal{A}_{jkl)} ~=~ 0 \quad , \tag{3.1}$$

$$\overline{D}_+^j\mathcal{A}_i ~-~ \tfrac{1}{2}\delta_i{}^j\overline{D}_+^k\mathcal{A}_k ~=~ C^{jk}\overline{D}_+^l\mathcal{A}_{ikl} \quad , \quad C_{m(i|}\overline{D}_+^m\mathcal{A}_{|jkl)} ~=~ 0 \quad . \tag{3.2}$$

$$C^{ij}D_{+i}\mathcal{B}_{=j} ~=~ 0 \quad , \quad \overline{D}_+^i\mathcal{B}_{=i} ~=~ 0 \quad . \tag{3.3}$$

These kinematic constraints may be compared with those of the (4,0) RHM obtainable from Table I. Examination of the first two lines above reveals that these equations are the (4,0) chiral projection of one of the special cases of the "generalized 2D, N = 4 tensor multiplet formalism" proposed by Ketov [13]. The component fields are obtained via the following projections

$$A_i ~\equiv~ \mathcal{A}_i| \quad , \quad A_{ijk} ~\equiv~ \mathcal{A}_{ijk}| \quad , \quad \overline{\pi}^- ~\equiv~ -\tfrac{1}{2}C^{ij}D_{+i}\mathcal{A}_j| \quad , \quad \rho^- ~\equiv~ \tfrac{1}{2}\overline{D}_+^i\mathcal{A}_i| \quad ,$$

$$\overline{\lambda}^-{}_{ij} ~\equiv~ \tfrac{1}{2}D_{+(i}\mathcal{A}_{j)}| \quad , \quad \rho^-{}_i{}^j ~\equiv~ [\,\overline{D}_+^j\mathcal{A}_i - \tfrac{1}{2}\delta_i{}^j\overline{D}_+^k\mathcal{A}_k\,]| \quad , \quad P_{\ddagger i} ~\equiv~ \tfrac{1}{2}\overline{D}_+^jD_{+(i}\mathcal{A}_{j)}| \quad ,$$

$$P_{=\, i} ~\equiv~ [\,\mathcal{B}_{=\, i} ~+~ i\partial_=\mathcal{A}_i\,]| \quad , \quad \overline{\chi}^+{}_{ij} ~\equiv~ \tfrac{1}{2}[\,D_{+(i|}(\,\mathcal{B}_{=|j)} ~+~ i\tfrac{1}{2}\partial_=\mathcal{A}_{|j)}\,)\,]| \quad ,$$

$$\beta^+{}_i{}^j ~\equiv~ [\,\overline{D}_+^j(\,\mathcal{B}_{=i} ~+~ i\tfrac{1}{2}\partial_=\mathcal{A}_i\,) ~-~ i\tfrac{1}{4}\delta_i{}^j\overline{D}_+^k\mathcal{A}_k\,]| \quad ,$$

$$B_{ijk} ~\equiv~ [\,C_{l\,(i|}\overline{D}_+^lD_{+|j|}\mathcal{B}_{=|k)} ~-~ \partial_\ddagger\partial_=\mathcal{A}_{ijk}\,]| \quad . \tag{3.4}$$

We do not explicitly write the supersymmetry variations of these fields since these are easily calculated from $\delta_Q ~=~ \epsilon^{+i}D_{+i} ~+~ \overline{\epsilon}^+{}_i\overline{D}_+{}^i$. (See the appendix.)

There are two separate superinvariants that we can form utilizing the fields in (3.4). This should be expected since similar structures existed for the RHM which was our starting point. The first supersymmetrically invariant Lagrangian is of the form,

$$
\begin{aligned}
\mathcal{L}_1 = & (\partial_{=} A_i)(\partial_{\ddagger} \overline{A}^i) + i\tfrac{1}{2}\overline{\pi}^- \partial_{=} \pi^- + i\tfrac{1}{2}\overline{\rho}^- \partial_{=} \rho^- \\
& - \tfrac{1}{6}(\partial_{=} A_{ijk})(\partial_{\ddagger} \overline{A}^{ijk}) + i\tfrac{3}{2}\overline{P}_{\ddagger}{}^i (\partial_{=} A_i) - i\tfrac{3}{2} P_{\ddagger i} (\partial_{=} \overline{A}^i) \\
& - i\tfrac{3}{4}\overline{\lambda}^-{}_{ij} \partial_{=} \lambda^{-ij} - i\tfrac{3}{4}\overline{\rho}^-{}_i{}^j \partial_{=} \rho^-{}_j{}^i \quad .
\end{aligned} \tag{3.5}
$$

A second supersymmetrically invariant Lagrangian takes the form

$$
\begin{aligned}
\mathcal{L}_2 = & \tfrac{3}{2}(\overline{P}_{\ddagger}{}^i P_{= i} + \overline{P}_{=}{}^i P_{\ddagger i}) - \tfrac{1}{12}(\overline{B}^{ijk} A_{ijk} + \overline{A}^{ijk} B_{ijk}) \\
& - \tfrac{3}{4}(\overline{\rho}^-{}_i{}^j \beta^+{}_j{}^i - \rho^-{}_i{}^j \overline{\beta}^+{}_j{}^i) - \tfrac{3}{4}(\lambda^{-ij} \overline{\chi}^+{}_{ij} - \overline{\lambda}^-{}_{ij} \chi^{+ij}) \\
& + \tfrac{1}{6}(\partial_{=} A_{ijk})(\partial_{\ddagger} \overline{A}^{ijk}) - i\tfrac{3}{2}\overline{P}_{\ddagger}{}^i (\partial_{=} A_i) + i\tfrac{3}{2} P_{\ddagger i} (\partial_{=} \overline{A}^i) \\
& + i\tfrac{3}{4}\overline{\lambda}^-{}_{ij} \partial_{=} \lambda^{-ij} + i\tfrac{3}{4}\overline{\rho}^-{}_i{}^j \partial_{=} \rho^-{}_j{}^i \quad .
\end{aligned} \tag{3.6}
$$

Using the Lagangian $\mathcal{L}^{(4,0)}_{\rm OHM} \equiv \mathcal{L}_1 + \mathcal{L}_2$, we find the final form of the action

$$
\begin{aligned}
\mathcal{S}^{(4,0)}_{\rm OHM} = \int d^2\sigma \Big[& (\partial_{=} A_i)(\partial_{\ddagger} \overline{A}^i) + i\tfrac{1}{2}\overline{\pi}^- \partial_{=} \pi^- + i\tfrac{1}{2}\overline{\rho}^- \partial_{=} \rho^- \\
& + \tfrac{3}{2}(\overline{P}_{\ddagger}{}^i P_{= i} + \overline{P}_{=}{}^i P_{\ddagger i}) - \tfrac{1}{12}(\overline{B}^{ijk} A_{ijk} + \overline{A}^{ijk} B_{ijk}) \\
& - \tfrac{3}{4}(\overline{\rho}^-{}_i{}^j \beta^+{}_j{}^i - \rho^-{}_i{}^j \overline{\beta}^+{}_j{}^i) - \tfrac{3}{4}(\lambda^{-ij} \overline{\chi}^+{}_{ij} - \overline{\lambda}^-{}_{ij} \chi^{+ij}) \Big] \quad .
\end{aligned} \tag{3.7}
$$

The component fields uniquely fix the unconstrained pre-potentials that describe the two multiplets. The first pre-potential is given by a dimension minus-one superfield $\Psi_{=\, i}$ (containing the fields in (2.9)) and the second pre-potential is a dimension zero superfield $\Psi_{==\, ijk}$ (containing the fields in (2.10)).

IV. (4,4) Relaxed Hypermultiplet → RADIO → (4,0) MSM-III Theory

Thus far, we have solved half of the long standing problem of finding an off-shell formulation whose on-shell limit contains the hypermultiplet as first described by Fayet. Our 2D (4,0) OHM theory satisfies this constraint. The solution to the complete problem of an off-shell 2D, N = 4 hypermultiplet requires the discovery of an additional 2D (4,0) minus spinor multiplet. The OHM theory in (2.9) and (2.10) is the analog of the RHM in (2.5) and (2.6). What is required is another version of the NM^{2}SM multiplet. Surprisingly, application of the Klein flip operator to (2.5) and (2.6) does <u>not</u> give the required version the NM^{2}SM multiplet.

Another 1D duality-type transformation is the automorphic duality (AD) map. Upon reducing the multiplets in (2.7) and (2.8) to one dimension, we can use the AD map upon the multiplets. When this is done appropriately, we arrive at a different 2D (4,0) NM^{2}SM multiplet. The spectrum of the component fields is contained in three multiplets. The first multiplet is composed of 4 - 4 fields

$$(\overline{\pi}^{+},\, \rho^{+},\, F_i) \tag{4.1}$$

a second composed of 12 - 12 component fields

$$(\overline{\lambda}^{+}{}_{ij},\, \rho^{+}{}_i{}^j,\, F_{ijk},\, G_i\,) \quad , \tag{4.2}$$

and a third multiplet also composed of the 12 - 12 component fields,

$$(K_i,\, K_{ijk},\, \overline{\chi}^{-}{}_{ij},\, \beta^{-}{}_i{}^j\,) \quad . \tag{4.3}$$

The first of these multiplets is actually one of the minimal minus spinor multiplets (SM-III) described previously [14]. The latter multiplets are completely auxiliary and are a minus spinor multiplet and scalar multiplet respectively.

These multiplets have a number of algebraically independent field strength superfields; $(\,\overline{\Pi}^{+},\, \Upsilon^{+}\,)$, $(\,\overline{\Pi}^{+}{}_{ij},\, \Upsilon^{+}{}_i{}^j\,)$ and $(\,\mathcal{H}_i,\, \mathcal{H}_{ijk})$. These field strengths satisfy a number of kinematic constraints,

$$\begin{gathered}
D_{+i}\overline{\Pi}^{+} \;=\; \overline{D}_{+}{}^{i}\Upsilon^{+} \;=\; D_{+i}\Upsilon^{+} \;-\; C_{ij}\overline{D}_{+}{}^{j}\overline{\Pi}^{+} \;=\; 0 \quad , \\
D_{+i}\overline{\Pi}^{+}{}_{jk} \;=\; \overline{D}_{+}{}^{i}\Upsilon^{+}{}_j{}^k \;=\; 0 \quad , \quad C_{l\,j}D_{+i}\Upsilon^{+}{}_k{}^l \;=\; -\,C_{l\,i}\overline{D}_{+}{}^{l}\overline{\Pi}^{+}{}_{jk} \quad , \\
C^{ij}D_{+i}\mathcal{H}_j \;=\; \overline{D}_{+}{}^{i}\mathcal{H}_i \;=\; 0 \quad , \\
D_{+i}\mathcal{H}_{jkl} \;=\; C_{i(j|}D_{+|k|}\mathcal{H}_{|l)} \quad , \quad \overline{D}_{+}{}^{i}\mathcal{H}_{jkl} \;=\; \delta_{(j|}{}^{i}\overline{D}_{+}{}^{m}\mathcal{H}_{|k|}C_{|l)m} \quad .
\end{gathered} \tag{4.4}$$

Similarly, the spectrum of component fields can be defined via projection as

$$\overline{\pi}^{+} \;\equiv\; \overline{\Pi}^{+}| \quad , \quad \rho^{+} \;\equiv\; \Upsilon^{+}| \quad , \quad F_i \;\equiv\; C_{ij}\overline{D}_{+}{}^{j}\overline{\Pi}^{+}| \quad ,$$

for the 4 - 4 multiplet and

$$\begin{gathered}
\rho^{+}{}_i{}^j \;\equiv\; \Upsilon^{+}{}_i{}^j| \quad , \qquad \overline{\lambda}^{+}{}_{ij} \;\equiv\; \overline{\Pi}^{+}{}_{ij}| \quad , \\
F_{ijk} \;\equiv\; \tfrac{1}{4}C_{l\,(i|}\overline{D}_{+}{}^{l}\overline{\Pi}^{+}{}_{|jk)}| \quad , \quad G_i \;\equiv\; \tfrac{1}{3}[\,D_{+j}\Upsilon^{+}{}_i{}^j\,]| \quad ,
\end{gathered} \tag{4.5}$$

for one of the 12 - 12 component multiplets and

$$K_i \;\equiv\; \mathcal{H}_i| \quad , \quad K_{ijk} \;\equiv\; \mathcal{H}_{ijk}| \quad , \quad \overline{\chi}^{-}{}_{ij} \;\equiv\; \tfrac{1}{2}D_{+(i}\mathcal{H}_{j)}| \quad , \quad \beta^{-}{}_i{}^j \;\equiv\; \overline{D}_{+}{}^{j}\mathcal{H}_i| \quad , \tag{4.6}$$

for the remaining 12 - 12 component multiplet.

Once again there are two separate superinvariants that we can form utilizing the fields in (4.4) and (4.5). The first supersymmetrically invariant Lagrangian is of the form,

$$\widehat{\mathcal{L}}_1 \;=\; i\tfrac{1}{2}\overline{\pi}^{+}\partial_{\ddagger}\pi^{+} \;+\; i\tfrac{1}{2}\overline{\rho}^{+}\partial_{\ddagger}\rho^{+} \;+\; \tfrac{1}{4}\overline{F}^{i}F_{i} \quad . \tag{4.7}$$

A second supersymmetrically invariant Lagrangians takes the form

$$\begin{aligned}\widehat{\mathcal{L}}_2 \;=&\; \tfrac{3}{2}(\, \overline{G}^{i}K_{i} \;+\; \overline{K}^{i}G_{i}\,) \;-\; \tfrac{1}{12}(\, \overline{K}^{ijk}F_{ijk} \;+\; \overline{F}^{ijk}K_{ijk}\,) \\ &-\; \tfrac{3}{4}(\, \lambda^{+ij}\overline{\chi}^{-}{}_{ij} \;-\; \overline{\lambda}^{+}{}_{ij}\chi^{-ij}\,) \;-\; \tfrac{3}{4}(\, \overline{\rho}^{+}{}_{i}{}^{j}\beta^{-}{}_{j}{}^{i} \;-\; \rho^{+}{}_{i}{}^{j}\overline{\beta}^{-}{}_{j}{}^{i}\,) \quad .\end{aligned} \tag{4.8}$$

We take the final Lagrangian to be a sum $\mathcal{L}^{(4,0)}_{\mathrm{NM^2SM-III}} \equiv \widehat{\mathcal{L}}_1 + \widehat{\mathcal{L}}_2$ and this yields a component action given by

$$\begin{aligned}\mathcal{S}^{(4,0)}_{\mathrm{NM^2SM-III}} \;=\; \int d^2\sigma \Big[\; & i\tfrac{1}{2}\overline{\pi}^{+}\partial_{\ddagger}\pi^{+} \;+\; i\tfrac{1}{2}\overline{\rho}^{+}\partial_{\ddagger}\rho^{+} \;+\; \tfrac{1}{4}\overline{F}^{i}F_{i} \\ &+\; \tfrac{3}{2}(\, \overline{G}^{i}K_{i} \;+\; \overline{K}^{i}G_{i}\,) \;-\; \tfrac{1}{12}(\, \overline{K}^{ijk}F_{ijk} \;+\; \overline{F}^{ijk}K_{ijk}\,) \\ &-\; \tfrac{3}{4}(\, \lambda^{+ij}\overline{\chi}^{-}{}_{ij} \;-\; \overline{\lambda}^{+}{}_{ij}\chi^{-ij}\,) \;-\; \tfrac{3}{4}(\, \overline{\rho}^{+}{}_{i}{}^{j}\beta^{-}{}_{j}{}^{i} \;-\; \rho^{+}{}_{i}{}^{j}\overline{\beta}^{-}{}_{j}{}^{i}\,) \;\Big] \quad .\end{aligned} \tag{4.9}$$

Once again the structure of the component fields fixes the pre-potential that describe this theory. The fields in (4.1) are contained in a dimension minus one pre-potential $\Psi_{=\!=i}$. The fields of (4.2) are contained in two dimension minus one pre-potentials $\widehat{\Psi}_{=\!=i}$ and $\widehat{\Psi}_{=\!=\,ijk}$. Finally the fields of (4.3) are contained in two chiral dimension one-half pre-potentials $\Psi_{-\,ij}$ and $\widehat{\Psi}_{-\,i}{}^{j}$.

V. 2D (4,4) Off-shell OHM Theory

It should be obvious that what we have derived in the last two sections are the two different chiral parts of a single 2D (4,4) theory. Our method of derivation, since it is actually one-dimensional, treats the two chiral parts separately. The results of these derivations must be integrated together into a single theory. We begin by noting that an action of the form

$$\mathcal{S}^{(4,4)}_{\mathrm{OHM}} \;=\; \mathcal{S}^{(4,0)}_{\mathrm{OHM}} \;+\; \mathcal{S}^{(4,0)}_{\mathrm{NM^2SM-III}} \quad , \tag{5.1}$$

contains all the degrees of freedom to describe the full (4,4) theory. Furthermore, if we interchange *all* +-signs with −-signs and vice-versa, we obtain the same action. So the sum in (5.1) has, in addition to (4,0) supersymmetry, a symmetry under parity reflections. This is exactly the necessary and sufficient condition of a $(p,0)$ theory to imply the existence of a full (p,p) supersymmetry.

The derivation of the form of the (0,4) supersymmetry variations can be done as follows. We first note that the complete spectrum of component fields is given in (2.9), (2.10), (4,1), (4.2) and (4.3). We next assert the existence of (0,4) operators D_{-i} and $\overline{D}_{-}{}^{i}$ acting on all component fields such that $D_{-i}\mathcal{S}^{(4,4)}_{\rm OHM} = 0$. The realization of D_{-i} on all fields is determined by taking the realization of D_{+i} on all fields and replacing <u>all</u> plus signs by minus signs and vice-versa.

Now having described how the (0,4) supersymmetry operators are derived, in principle we can derive the realization of the full (4,4) supersymmetry algebra. Without giving explicit results, we note the form it takes is given by

$$
\begin{aligned}
&[\, D_{+i} \,,\, \overline{D}_{+}{}^{j} \,\} \;=\; i2\delta_i{}^j\partial_{\ddagger} \quad , \quad [\, D_{-i} \,,\, \overline{D}_{-}{}^{j} \,\} \;=\; i2\delta_i{}^j\partial_{=} \quad , \quad [\, D_{\pm i} \,,\, D_{\pm j} \,\} \;=\; 0 \quad , \\
&\qquad\qquad [\, D_{+i} \,,\, D_{-j} \,\} \;=\; i2\,\mathcal{Z}^{(1)}{}_{ij} \quad , \quad [\, D_{+i} \,,\, \overline{D}_{-}{}^{j} \,\} \;=\; i2\,\mathcal{Z}^{(2)}{}_{i}{}^{j} \quad . \qquad (5.2)
\end{aligned}
$$

In other words, the full (4,4) algebra requires the presence of two complex central charges $\mathcal{Z}^{(1)}{}_{ij}$ and $\mathcal{Z}^{(2)}{}_{i}{}^{j}$.

VI. Summary

In this work we have used the RADIO technique to derive new results for off-shell realizations of 2D (4,0) supersymmetry. This powerfully demonstrates the surprising capabilities inherent in the 1D formulation of higher dimensional theories. The fact, that we were able to start from the relaxed hypermultiplet and derive results for the Fayet hypermultiplet suggests that there are many such connections to even more non-minimal theories. In fact, our past experience [14] with the minimal off-shell formulations taught us to expect the (4,0) theories to come in several distinct versions. The multiplet in (2.3) and (2.4) is a non-minimal extension of the SM-II theory (in the classification of (4,0) scalar multiplets of [14]). The scalar multiplet in (2.9) and (2.10) is a non-minimal extension of SM-III. Similarly, the spinor multiplet of (2.7) and (2.8) is a non-minimal extension of the MSM-I and/or MSM-II theories.

The occurrence of these non-minimal representations also has possible implications for the ADHM non-linear σ-model construction [15]. In our first investigation of the manifest realization of (4,0) supersymmetry in this class of models, we showed that it was not possible to construct such a theory using only <u>minimal</u> and manifest (4,0) supersymmetry representations. This led some [16] to the premature conclusion that it is not possible to construct off-shell (4,0) ADHM models utilizing a finite number of auxiliary fields. This conclusion was premature since no use of non-minimal representations was investigated.

We have long suspected that the apparently infinite number of auxiliary fields that seemed to be required by harmonic superspace formulations is an illusion. Our present work supports this view. If this view is always true, then the harmonic superspace approach is simply an expensive luxury (i.e. $112 << \infty$). It will be important to derive our present results from harmonic superspace since this may permit the latter to be used as a method of derivation of off-shell formulations with a finite numbers of auxiliary fields in other theories (i.e. N = 4 supersymmetric Yang-Mills theory, etc.). The harmonics introduced in the harmonic superspace approach are likely to ultimately be analogous to the extra "Vainberg coordinate" [17] introduced by Novikov and Witten [18] for the description of the WZNW term of QCD. The Vainberg construction is a convenient, not essential, method for describing the WZNW action. Similarly, harmonic superspace may be a convenient way to derive off-shell formulations. We propose that there should exist projection operators for harmonic superspace that can be used to recover distinct off-shell theories with a finite number of auxiliary fields.

Appendix A: D-Algebra Realization and Alternate Formulations

In this appendix, we give the complete and explicit realization of the D_+-operators on all fields. The realization of the D_--operators is uniformly obtained by interchanging <u>all</u> plus and minus signs below.

$$
\begin{gathered}
D_{+i} A_j ~=~ - C_{ij} \overline{\pi}{}^- ~+~ \overline{\lambda}{}^-{}_{ij} \quad , \quad \overline{D}_+{}^i A_j ~=~ \delta_j{}^i \rho^- ~+~ \rho^-{}_j{}^i \quad , \\
D_{+i} \overline{\lambda}{}^-{}_{jk} ~=~ 0 \quad , \quad \overline{D}_+{}^i \rho^-{}_j{}^k ~=~ 0 \quad , \quad D_{+i} \overline{\pi}{}^- ~=~ 0 \quad , \quad \overline{D}_+{}^i \rho^- ~=~ 0 \quad , \\
D_{+i} \rho^- ~=~ - [\, 3 P_{\ddagger i} ~+~ i 2 \partial_{\ddagger} A_i \,] \quad , \quad \overline{D}_+{}^i \overline{\pi}{}^- ~=~ C^{ij} [\, 3 P_{\ddagger j} ~+~ i 2 \partial_{\ddagger} A_j \,] \quad , \\
D_{+i} \rho^-{}_j{}^k ~=~ 2 [\, \delta_i{}^k P_{\ddagger j} ~-~ \tfrac{1}{2} \delta_j{}^k P_{\ddagger i} \,] ~+~ i \tfrac{2}{3} C^{kl} \partial_{\ddagger} A_{ijl} \quad , \\
\overline{D}_+{}^i \overline{\lambda}{}^-{}_{jk} ~=~ \delta_{(j|}{}^i P_{\ddagger |k)} ~-~ i \tfrac{2}{3} C^{il} \partial_{\ddagger} A_{jkl} \quad , \\
D_{+i} P_{\ddagger j} ~=~ - i \tfrac{2}{3} \partial_{\ddagger} \overline{\lambda}{}^-{}_{ij} \quad , \quad \overline{D}_+{}^i P_{\ddagger j} ~=~ - i \tfrac{2}{3} \partial_{\ddagger} \rho^-{}_j{}^i \quad , \\
D_{+i} A_{jkl} ~=~ \tfrac{1}{2} C_{i(j|} \overline{\lambda}{}^-{}_{|kl)} \quad , \quad \overline{D}_+{}^i A_{jkl} ~=~ \tfrac{1}{2} \delta_{(j|}{}^i \rho^-{}_{|k|}{}^m C_{|l)m} \quad ,
\end{gathered}
\qquad (A.1)
$$

$$
\begin{gathered}
D_{+i} P_{= j} ~=~ \overline{\chi}{}^+{}_{ij} ~+~ i \tfrac{1}{2} \partial_{=} \overline{\lambda}{}^-{}_{ij} ~-~ i C_{ij} \partial_{=} \overline{\pi}{}^- \quad , \quad D_{+i} \overline{\chi}{}^+{}_{jk} ~=~ 0 \\
\overline{D}_+{}^i P_{= j} ~=~ \beta^+{}_j{}^i ~+~ i \tfrac{1}{2} \partial_{=} \rho^-{}_j{}^i ~+~ i \delta_j{}^i \partial_{=} \rho^- \quad , \quad \overline{D}_+{}^i \beta^+{}_j{}^k ~=~ 0 \quad , \\
\overline{D}_+{}^i \overline{\chi}{}^+{}_{jk} ~=~ -~ \tfrac{2}{3} \delta_{(j|}{}^i \partial_{\ddagger} \partial_{=} A_{|k)} ~+~ i \tfrac{1}{2} \delta_{(j|}{}^i [\, \partial_{=} P_{\ddagger |k)} ~-~ \tfrac{4}{3} \partial_{\ddagger} P_{= |k)} \,] ~-~ \tfrac{1}{3} C^{il} B_{jkl} \quad , \\
D_{+i} \beta^+{}_j{}^k ~=-~ \tfrac{4}{3} \partial_{\ddagger} \partial_{=} [\, \delta_i{}^k A_j ~-~ \tfrac{1}{2} \delta_j{}^k A_i \,] ~+~ i \delta_i{}^k [\, \partial_{=} P_{\ddagger j} ~-~ \tfrac{4}{3} \partial_{\ddagger} P_{= j} \,] \\
-~ i \tfrac{1}{2} \delta_j{}^k [\, \partial_{=} P_{\ddagger i} ~-~ \tfrac{4}{3} \partial_{\ddagger} P_{= i} \,] ~+~ \tfrac{1}{3} C^{kl} B_{ijl} \quad ,
\end{gathered}
$$

$$D_{+i}B_{jkl} \;=\; iC_{i(j|}\partial_{\ddagger}\overline{\chi}^{+}{}_{|kl)} \quad , \quad \overline{D}_{+}{}^{i}B_{jkl} \;=\; i\delta_{(j|}{}^{i}\partial_{\ddagger}\beta^{+}{}_{|k|}{}^{m}C_{|l)m} \quad , \qquad (A.2)$$

$$D_{+i}\overline{\pi}^{+} \;=\; 0 \quad , \quad \overline{D}_{+}{}^{i}\rho^{+} \;=\; 0 \quad , \quad D_{+i}\rho^{+} \;=\; F_{i} \quad , \quad \overline{D}_{+}{}^{i}\overline{\pi}^{+} \;=\; -C^{ij}F_{j} \quad ,$$
$$D_{+i}F_{j} \;=\; i2C_{ij}\partial_{\ddagger}\overline{\pi}^{+} \quad , \quad \overline{D}_{+}{}^{i}F_{j} \;=\; -i2\delta_{j}{}^{i}\partial_{\ddagger}\rho^{+} \quad , \qquad (A.3)$$

$$D_{+i}\rho^{+}{}_{j}{}^{k} \;=\; 2(\,\delta_{i}{}^{k}G_{j} \;-\; \tfrac{1}{2}\delta_{j}{}^{k}G_{i}\,) \;+\; \tfrac{2}{3}C^{kl}F_{ijl} \quad , \quad \overline{D}_{+}{}^{i}\rho^{+}{}_{j}{}^{k} \;=\; 0 \quad ,$$
$$D_{+i}\overline{\lambda}^{+}{}_{jk} \;=\; 0 \quad , \quad \overline{D}_{+}{}^{i}\overline{\lambda}^{+}{}_{jk} \;=\; \delta_{(j}{}^{i}G_{k)} \;-\; \tfrac{2}{3}C^{il}F_{jkl} \quad ,$$
$$D_{+i}G_{j} \;=\; -i\tfrac{2}{3}\partial_{\ddagger}\overline{\lambda}^{+}{}_{ij} \quad , \quad \overline{D}_{+}{}^{i}G_{j} \;=\; -i\tfrac{2}{3}\partial_{\ddagger}\rho^{+}{}_{j}{}^{i} \quad ,$$
$$D_{+i}F_{jkl} \;=\; i\tfrac{1}{2}C_{i(j|}\partial_{\ddagger}\overline{\lambda}^{+}{}_{|kl)} \quad , \quad \overline{D}_{+}{}^{i}F_{jkl} \;=\; i\tfrac{1}{2}\delta_{(j|}{}^{i}\partial_{\ddagger}\rho^{+}{}_{|k|}{}^{m}C_{|l)\,m} \quad . \qquad (A.4)$$

$$D_{+i}K_{j} \;=\; \overline{\chi}^{-}{}_{ij} \quad , \quad \overline{D}_{+}{}^{i}K_{j} \;=\; \beta^{-}{}_{j}{}^{i} \quad , \quad D_{+i}\overline{\chi}^{-}{}_{jk} \;=\; 0 \quad , \quad \overline{D}_{+}{}^{i}\beta^{-}{}_{j}{}^{k} \;=\; 0 \quad ,$$
$$D_{+i}\beta^{-}{}_{j}{}^{k} \;=\; -i\tfrac{4}{3}(\,\delta_{i}{}^{k}\partial_{\ddagger}K_{j} \;-\; \tfrac{1}{2}\delta_{j}{}^{k}\partial_{\ddagger}K_{i}\,) \;+\; i\tfrac{1}{3}C^{kl}\partial_{\ddagger}K_{ijl} \quad ,$$
$$\overline{D}_{+}{}^{i}\overline{\chi}^{-}{}_{jk} \;=\; -i\tfrac{2}{3}\delta_{(j}{}^{i}\partial_{\ddagger}K_{k)} \;-\; i\tfrac{1}{3}C^{il}\partial_{\ddagger}K_{jkl} \quad ,$$
$$D_{+i}K_{jkl} \;=\; C_{i(j|}\overline{\chi}^{-}{}_{|kl)} \quad , \quad \overline{D}_{+}{}^{i}K_{jkl} \;=\; \delta_{(j|}{}^{i}\beta^{-}{}_{|k|}{}^{m}C_{|l)\,m} \quad . \qquad (A.5)$$

Finally we note that the formulation described above lends itself to other re-formulations of the off-shell 2D (4,4) OHM theory. Under the following field re-definitions $P_{\ddagger i} \;\to\; i\partial_{\ddagger}P_{i}$, $P_{=i} \;\to\; i\Box^{-1}\partial_{=}B_{i}$ a different local theory exists. It has three bosonic field strengths $\mathcal{A}_{i}$, $\mathcal{P}_{i}$ and $\mathcal{P}_{ijk}$ containing the component fields

$$(A_{i},\, \overline{\pi}^{-},\, \rho^{-}\,) \quad , \quad (A_{ijk},\, P_{i},\, \overline{\lambda}^{-}{}_{ij},\, \rho^{-}{}_{i}{}^{j}\,) \quad , \qquad (A.6)$$

and as well spinorial field strength tensors $\mathcal{C}^{+}{}_{ij}$ and $\mathcal{B}^{+}{}_{i}{}^{j}$ containing the component field content,

$$(\overline{\chi}^{+}{}_{ij},\, \beta^{+}{}_{i}{}^{j},\, B_{ijk},\, B_{i}\,) \quad . \qquad (A.7)$$

The component form of the action is obtained by applying the field re-definition described above to (3.7). This version of the (4,0) OHM has a structure like that of (4.1), (4.2) and (4.3) where the first multiplet in (A.6) replaces the minus spinor multiplet. The spectrum of fields in (4.1) - (4.3) together with those of (A.6) and (A.7) has the interesting feature that they can all be interpreted as arising from 2D, N = 2 chiral superfields: Φ_{i}, $\widehat{\Phi}_{i}$, $\widetilde{\Phi}_{i}$, $\widehat{\Phi}_{ijk}$ and $\widetilde{\Phi}_{ijk}$.

One other alternative formulation utilizes the re-definitions $B_{ijk} \;\to\; -i\partial_{\ddagger}P_{=ijk}$, $A_{ijk} \;\to\; -i\Box^{-1}\partial_{=}P_{\ddagger ijk}$. Under this re-definition, the multiplets of (2.9) and (2.10) are replaced by

$$(A_{i},\, \overline{\pi}^{-},\, \rho^{-},\, \overline{\lambda}^{-}{}_{ij},\, \rho^{-}{}_{i}{}^{j},\, P_{\ddagger i},\, P_{\ddagger ijk}\,) \;\; , \;\; (P_{=i},\, P_{=ijk},\, \overline{\chi}^{+}{}_{ij},\, \beta^{+}{}_{i}{}^{j}\,) \quad . \qquad (A.8)$$

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